THE NUMBER ZERO

❑ PROPERTIES OF ZERO

$\boldsymbol{N + 0 = N}$ for any number N.

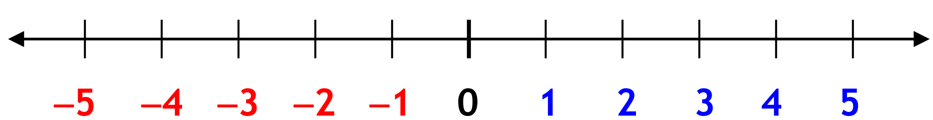
$\boldsymbol{N \cdot 0 = 0}$ for any number N.

The ***opposite*** of 0 is 0.

0 has no ***reciprocal***. [If it did, it would be $\frac{1}{0}$, which is *undefined*, as described on Page 2.]

❑ THE NUMBER LINE

Here's a picture of the positive and negative numbers (and **zero**) called the ***number line***. It is also called the real line because all real numbers can be located somewhere on the line. So, even though I've labeled just a few integers on the line, all kinds of numbers, like $\frac{2}{3}$, π, $-\sqrt{7}$, 1.90977, and –2.37777. . . are all there somewhere.



The ***positive*** numbers are to the right of zero, while the ***negative*** numbers are to the left of zero. The positive number 4 could be written +4, but it's not necessary — we know that the plus sign is understood. By the way, 0 is neither positive nor negative; it's considered neutral, and is called the ***origin*** of the number line.

❑ DIVISION AND THOSE PESKY ZEROS

One of the most important facts in all of mathematics is that the denominator (bottom) of a fraction can NEVER be zero.

Sometimes this is phrased

"Never divide by zero."

What's the big deal? Why can't we do it?

This is the result of dividing by zero.

Recall from the discussion of dividing signed numbers that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

We don't have to blindly accept the fact that you should never divide by 0. Let's put zeros in fractions and see what happens.

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work (confirm this yourself).

Zero on the Bottom

Now let's put a zero on the bottom (but not the top) and see what happens:

$$\frac{9}{0} = ???$$

Your first guess might be 0; let's check it out:

$$\frac{9}{0} = 0 \text{ would be true only if } 0 \cdot 0 = 9\text{, which it isn't.}$$

How about we try an answer of 9?

$$\frac{9}{0} = 9 \text{ which is checked by seeing if } 9 \cdot 0 = 9\text{. Sorry.}$$

Could the answer be π? Nope; $\pi \cdot 0 = 0$, not 9.

In fact, <u>any</u> number we conjure up as a potential answer will have to multiply with 0 to make a product of 9. But this is <u>impossible</u>, since any number times 0 is always 0, never 9. In short, <u>*no number* in the whole world</u> will work as the answer to this division problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0: $\frac{0}{0} = 0$ which seems reasonable, since $0 \cdot 0 = 0$.

Let's try an answer of 5: $\frac{0}{0} = 5$, which amazingly! — works out too, because $5 \cdot 0 = 0$.

Could π possibly work? $\frac{0}{0} = \pi$ — still true, since $\pi \cdot 0$ is surely 0.

That's three answers for this division problem: 0, 5, and π.

Is there any end to this madness? Apparently not, since <u>any</u> number we think up will multiply with 0 to make a product of 0. In short, <u>*every number* in the whole world</u> will work in this division problem.

SUMMARY:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is <u>no answer</u> to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are <u>infinitely many answers</u> to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the two kinds of division problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

<u>DIVISION BY ZERO IS UNDEFINED!</u>

And one more summary,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

"Black holes are where God divided by zero."

Steven Wright

Note: The two division problems with zero in the denominator may be undefined, but they're undefined for different reasons. Your teacher may require you to understand this.

❑ *ZERO AS AN EXPONENT*

We've already learned that anything to the zero power is 1 (as long as it's not zero to the zero). We reached this conclusion after we learned that $2^0 = 1$, and figured it might be true for any base. Now we try to verify this; that is, what is x^0? Consider the expression

$$x^3x^0 \qquad \text{where we assume } x \neq 0.$$

To figure out the meaning of x^0, we can use the first law of exponents to calculate

$$x^3x^0 = x^{3+0} = x^3$$

That is,

$$x^3x^0 = x^3$$

Now "isolate" the x^0, since that's what we're trying to find the value of. We do this by dividing each side of the equation by x^3:

$$\frac{x^3x^0}{x^3} = \frac{x^3}{x^3},$$

which implies that

$$x^0 = 1,$$

and we're done:

Any number (except 0) raised to the zero power is 1.

EXAMPLE:

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the -187)

F. $-14^0 = \mathbf{-1}$ (the exponent in on the 14, not on the minus sign)